MATH2048 Honours Linear Algebra II

Midterm Examination 1

Please show all your steps, unless otherwise stated. Answer all **five** questions.

1. Let $V = M_{2\times 2}(\mathbb{R})$. Define $T : V \to \mathbb{R}$ by $T(A) = A_{11} - A_{22}$ and $U : P_1(\mathbb{R}) \to V$ by $U(p) = \begin{pmatrix} 0 & p(0) \\ -p(0) & p(1) \end{pmatrix}$.

   (a) Find a basis β for $N(T)$ and a basis γ for $R(U)$.

   (b) Find the dimensions of $N(T)$, $R(U)$, $N(T) \cap R(U)$ and $N(T) + R(U)$.

2. Consider the linear operator T defined by

$$T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$$
$$A \mapsto A - A^T$$

   Let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be the standard ordered basis for $M_{2\times 2}(\mathbb{R})$.

   (a) Find an ordered basis γ_1 for $N(T)$ and an ordered basis γ_2 for $R(T)$. Show that $\gamma = \gamma_1 \cup \gamma_2$ is an ordered basis for $M_{2\times 2}(\mathbb{R})$.

   (b) With the γ that you found in (a), find the matrices $A = [T]_\gamma$ and $B = [I_{M_{2\times 2}(\mathbb{R})}]_\gamma^\beta$. Represent $[T]_\beta$ by A and B using change of coordinates (No need to do the computation).

3. Consider the linear transformation

$$T : P(\mathbb{R}) \to P(\mathbb{R})$$
$$p(x) \mapsto p(x+1)$$

   Note that $P(\mathbb{R}) = \bigcup_{n=0}^{\infty} P_n(\mathbb{R})$. Let $T|_{P_n(\mathbb{R})} : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the restriction of T on $P_n(\mathbb{R})$. Let β_n be the standard ordered basis for $P_n(\mathbb{R})$ for non-negative integer n.

   (a) Prove that $T(\beta_n)$ is a basis for $P_n(\mathbb{R})$, that is $T|_{P_n(\mathbb{R})}$ is onto. Deduce that T is onto. (Hint: The Binomial Theorem $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$.)

   (b) Use (a) to show that $T|_{P_n(\mathbb{R})}$ is one-to-one. Deduce that T is one-to-one.

4. Let $C^\infty(\mathbb{R})$ be the vector space of all smooth real functions (infinitely differentiable) over $\mathbb{R}$. Let V be the subspace of $C^\infty(\mathbb{R})$ defined by:

$$V = \{f \in C^\infty(\mathbb{R}) : f(0) = f'(0) = \ldots = f^{(n)}(0) = 0\},$$

   where $f^{(j)}$ denotes the j-th derivative of f.

   (a) Define $\Psi : C^\infty(\mathbb{R}) \to \mathbb{R}^{n+1}$ by $\Psi(f) = (f(0), f'(0), \ldots, f^{(n)}(0))$. Show that Ψ is onto.

   (b) Define $\tilde{\Psi} : C^\infty(\mathbb{R})/V \to \mathbb{R}^{n+1}$ by $\tilde{\Psi}(f + V) = \Psi(f)$. Use (a) to show that $\tilde{\Psi}$ is an isomorphism, i.e. well-defined, linear and bijective.

5. Let U be a non-zero subspace of an infinite dimensional vector space V over F. Let $L \subset U$ be a basis of U. Using Zorn's lemma, show that L can be extended to a basis of V. Deduce that there exists a subspace W of V such that $V = U \oplus W$. Please explain your answer with all the details.